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The Energy of n Identical Bosons in a Finite Volume at $\mathcal{O}(L^{-7})$

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The volume dependence of the ground-state energy of n identical bosons with short-range interactions in a periodic spatial volume with sides of length L is calculated at order L^{-7} in the large volume expansion. This result will enable a refined determination of the $\pi^+\pi^+\pi^+$ interaction from lattice QCD calculations.

It is now well established that two-body interactions between hadrons can be studied with lattice QCD as the volume dependence of the energy spectrum of two hadrons is related to their scattering amplitude below inelastic thresholds [1, 2]. Recently, this method has been used to determine the $\pi^+\pi^+$ scattering length [3], $a_{\pi^+\pi^+}$, with $\sim 1\%$ precision with a $n_f = 2+1$ fully-dynamical mixed-action lattice QCD calculation. In order to extract the many-body interactions from lattice QCD calculations, the energy of multi-hadron states in a finite volume must be calculated with lattice QCD and combined with the known dependence of this energy on the many-body interactions. The ground state energy of a system of n identical bosons with short-range interactions in a cubic volume with sides of length L was recently computed at $\mathcal{O}(L^{-6})$ in the large-volume expansion [4]. The underlying motivation for that work, which builds upon the classic works of Refs. [5, 6, 7, 8], was to provide a way to determine the three-body interactions between π^+'s from lattice QCD calculations, which first enters at that order [4]. In Ref. [9], this result was used in conjunction with lattice QCD calculations of multi-pion systems to determine the interaction between three π^+'s for the first time. In order to refine the determination of the $\pi^+\pi^+\pi^+$ interaction, here we compute the contribution to the energy-shift of n identical bosons at $\mathcal{O}(L^{-7})$ in the large volume expansion. The energy-shift of three identical bosons in a finite-volume has been been computed recently in Ref. [10], and our $n = 3$ calculation agrees.

The ground-state energy of n identical bosons is calculated using standard Schrödinger perturbation theory, with a Hamiltonian, appropriate to the order we are working in the large volume expansion, of the form

$$
\begin{aligned}
H \ = \ & \sum_{\mathbf{k}} h^\dagger_{\mathbf{k}}\, h_{\mathbf{k}} \left(\frac{|\mathbf{k}|^2}{2M} - \frac{|\mathbf{k}|^4}{8M^3} \right) \\
& + \frac{1}{(2!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p}} h^\dagger_{\frac{\mathbf{Q}}{2}+\mathbf{k}} h^\dagger_{\frac{\mathbf{Q}}{2}-\mathbf{k}}\, h_{\frac{\mathbf{Q}}{2}+\mathbf{p}}\, h_{\frac{\mathbf{Q}}{2}-\mathbf{p}} \left(\frac{4\pi a}{M} + \frac{\pi a}{M}\left(ar - \frac{1}{2M^2}\right) \left(|\mathbf{k}|^2 + |\mathbf{p}|^2 \right) \right) \\
& + \frac{\eta_3(\mu)}{(3!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p},\mathbf{r},\mathbf{s}} h^\dagger_{\frac{\mathbf{Q}}{3}+\mathbf{k}} h^\dagger_{\frac{\mathbf{Q}}{3}+\mathbf{p}}\, h^\dagger_{\frac{\mathbf{Q}}{3}-\mathbf{k}-\mathbf{p}}\, h_{\frac{\mathbf{Q}}{3}+\mathbf{r}} h_{\frac{\mathbf{Q}}{3}+\mathbf{s}}\, h_{\frac{\mathbf{Q}}{3}-\mathbf{r}-\mathbf{s}} \ ,
\end{aligned}
\tag{1}
$$

where the operator $h_{\mathbf{k}}$ annihilates a π^+ with momentum $\mathbf{k}$ with unit amplitude. The divergences that arise at loop-level are regulated with dimensional regularization, and therefore the coefficients of the two-body interaction can be readily identified with the parameters describing the scattering amplitude: the scattering length, a, and the effective range, r ($p\cot\delta = -\frac{1}{a} + \frac{1}{2}rp^2 + \ ...$). The terms proportional to M^{-3} in eq. (1) describe the leading effects of relativity. Only the momentum independent three-body interaction, $\eta_3(\mu)$, is required at $\mathcal{O}(L^{-7})$. Our method of computation is equivalent to the pionless EFT describing low-energy nucleon-nucleon interactions, EFT($\not\pi$) [11, 12, 13] (when modified to describe systems with natural scattering lengths) and the method of pseudo-potentials used in our previous work. The divergences that occur in loop diagrams are renormalized order-by-order in the expansion, preserving the power counting, and hence the explicit dependence of the bare three-body coefficient on the renormalization scale, μ.

The calculation of the energy-shift of n identical bosons at $\mathcal{O}(L^{-7})$ due to the interactions defined in eq. (1) is straightforward but tedious. We will not delve into the details, referring the reader to our previous work [4] and that

of Ref. [10], and simply state the result. The energy-shift of the ground state is

$$
\begin{aligned}
E_0(n,L) \;=\; & \frac{4\pi\, a}{M\, L^3}\binom{n}{2}\Bigg\{1-\left(\frac{a}{\pi\, L}\right)\mathcal{I}+\left(\frac{a}{\pi\, L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right] \\
& \qquad -\left(\frac{a}{\pi\, L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+\left(5n^2-41n+63\right)\mathcal{K}\right] \\
& \qquad +\left(\frac{a}{\pi\, L}\right)^4\Big[\mathcal{I}^4-6\mathcal{I}^2\mathcal{J}+(4+n-n^2)\mathcal{J}^2+4(27-15n+n^2)\mathcal{I}\,\mathcal{K} \\
& \qquad\qquad +(14n^3-227n^2+919n-1043)\mathcal{L}\,\Big]\Bigg\} \\
& +\binom{n}{2}\frac{8\pi^2 a^3 r}{M\, L^6}\left[\,1\;+\;\left(\frac{a}{\pi\, L}\right)3(n-3)\mathcal{I}\,\right] \\
& +\binom{n}{3}\frac{1}{L^6}\left[\,\eta_3(\mu)\;+\;\frac{64\pi a^4}{M}\left(3\sqrt{3}-4\pi\right)\,\log\left(\mu L\right)\;-\;\frac{96a^4}{\pi^2 M}\,\mathcal{S}\,\right]\left[1\;-\;6\,\left(\frac{a}{\pi L}\right)\,\mathcal{I}\,\right] \\
& +\binom{n}{3}\left[\,\frac{192\; a^5}{M\pi^3 L^7}\left(\mathcal{T}_0\;+\;\mathcal{T}_1\; n\right)\;+\;\frac{6\pi a^3}{M^3 L^7}\,(n+3)\,\mathcal{I}\,\right]\;+\;\mathcal{O}\left(L^{-8}\right) \qquad . \qquad (2)
\end{aligned}
$$

where the geometric constants that enter are [1]

$$
\begin{aligned}
\mathcal{I} &= -8.9136329 & \qquad \mathcal{T}_0 &= -4116.2338 \\
\mathcal{J} &= 16.532316 & \mathcal{T}_1 &= 450.6392 \\
\mathcal{K} &= 8.4019240 & \mathcal{S}_{\rm MS} &= -185.12506 \\
\mathcal{L} &= 6.9458079 & & \qquad (3)
\end{aligned}
$$

and $\binom{n}{k}=n!/(n-k)!/k!$. The last term in the last bracket of eq. (2) is the leading relativistic contribution to the energy-shift. Deviations from the energy-shift of n-bosons computed with non-relativistic quantum mechanics arise only for three or more particles as the two-particle energy-shift has the same form when computed in non-relativistic quantum mechanics and in quantum field theory [1, 2]. In eq. (3), $\mathcal{S}_{\rm MS}$ is the value of the scheme-dependent quantity $\mathcal{S}$ in the Minimal Subtraction (MS) scheme that we have employed to renormalize the theory (a change in scheme results in a change in $\mathcal{S}$ and a compensating change in $\eta_3(\mu)$).[2] The $\mathcal{T}_i$ are renormalization scheme independent. Our result at $n=2$ agrees with large volume expansion of Ref. [1, 2], and at $n=3$ agrees with the previous computation by Shina Tan [10].

The renormalization-scale independent, but volume dependent, quantity

$$
\overline{\eta}_3^L = \eta_3(\mu)\;+\;\frac{64\pi a^4}{m}\left(3\sqrt{3}-4\pi\right)\,\log\left(\mu L\right)\;-\;\frac{96a^4}{\pi^2 m}\mathcal{S} \qquad (4)
$$

was determined in recent lattice QCD calculations [9]. It was found to be non-vanishing in systems of three, four and five π^+'s at pion masses of $m_\pi\sim 290$ and 350 MeV in a $\sim(2.5\ \mathrm{fm})^3$ volume, when extracted at $\mathcal{O}(L^{-6})$ in the

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1 The constants $\mathcal{I},\mathcal{J},\mathcal{K}$ were defined previously in Ref. [4], while the constant $\mathcal{L}$ is defined to be the integer-triplet sum

$$
\mathcal{L} \;=\; \sum_{\mathbf{n}\neq\mathbf{0}}\frac{1}{|\mathbf{n}|^8} \quad ,
$$

and is equal to $\mathcal{L}=\alpha_4$ in the notation of Ref. [10]. The constants $\mathcal{T}_{0,1}$ arise from combinations of up to three-loop diagrams, and involve three-, six- and nine-dimensional sums over integers, and can be written in terms of constants defined in Ref [10] plus one additional sum, S_1,

$$
\mathcal{T}_0+\mathcal{T}_1\; n \;=\; \frac{1}{4}\alpha_{1AA1}-\mathcal{I}\;\alpha_{1A1}+\frac{1}{2}(2n-9)\alpha_{2A1}+\frac{3}{4}(n-4)\alpha_{1B1}-\frac{1}{4}(7n-29)\mathcal{L}+2(n-3)S_1 \quad ,
$$

where

$$
S_1 \;=\; \sum_{\mathbf{n},\mathbf{j}\neq\mathbf{0}}\frac{1}{|\mathbf{n}|^2|\mathbf{j}|^4\,[\;|\mathbf{n}|^2+|\mathbf{n}+\mathbf{j}|^2]} \;=\; 92.42215 \quad .
$$

2 In the notation of Ref. [4], $\mathcal{S}_{\rm MS}=2\mathcal{Q}+\mathcal{R}$. The numerical value in eq. (3) corrects a minor error in $\mathcal{Q}$ in Ref. [4].

large volume expansion. Its size was found to be consistent with expectations based upon naive dimensional analysis, $\overline{\eta}_3^L \sim 1/(m_\pi f_\pi^4)$. Our result will allow for further refinement of such extractions.

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